

# The Center and Radius of the Regular Graph of Ideals <sup>\*†</sup>

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## Abstract

The regular graph of ideals of the commutative ring  $R$ , denoted by  $\Gamma_{reg}(R)$ , is a graph whose vertex set is the set of all non-trivial ideals of  $R$  and two distinct vertices  $I$  and  $J$  are adjacent if and only if either  $I$  contains a  $J$ -regular element or  $J$  contains an  $I$ -regular element. In this paper, it is proved that the radius of  $\Gamma_{reg}(R)$  equals 3. The central vertices of  $\Gamma_{reg}(R)$  are determined, too.

## 1. Introduction

We begin with recalling some definitions and notations on graphs. Let  $G$  be a simple graph. The *distance* between two vertices  $x$  and  $y$  of  $G$  is denoted by  $d(x, y)$ . A graph is said to be *connected* if there exists a path between any two distinct vertices. The *diameter* of a connected graph  $G$ , denoted by  $diam(G)$ , is the maximum distance between any pair of vertices of  $G$ . For any vertex  $x$  of a connected graph  $G$ , the *eccentricity* of  $x$ , denoted by  $e(x)$ , is the maximum of the distances from  $x$  to the other vertices of  $G$ . The set of vertices with minimal eccentricity is called the *center* of the graph, and this minimum eccentricity value is the *radius* of  $G$ . Let  $\Gamma$  be a digraph. An arc from a vertex  $x$  to another vertex  $y$  of  $\Gamma$  is denoted by  $x \longrightarrow y$ . Also we distinguish the *out-degree*  $d_{\Gamma}^{+}(v)$ , the number of edges leaving the vertex  $v$ , and the *in-degree*  $d_{\Gamma}^{-}(v)$ , the number of edges entering the vertex  $v$ . For more details about the standard terminology of graphs, see [6].

Unless otherwise stated, throughout this paper, all rings are assumed to be commutative Artinian rings with identity. We denote by  $Max(R)$  and  $Nil(R)$ , the set of all maximal ideals and the set of all nilpotent elements of  $R$ , respectively. The ring  $R$  is said to be *reduced* if  $Nil(R) = (0)$ . Also, the set of all zero-divisors of an  $R$ -module  $M$  is

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denoted by  $Z(M)$ . An element  $r$  in the ring  $R$  is called  $M$ -regular if  $r \notin Z(M)$ . For every ideal  $I$  of  $R$ , the *annihilator* of  $I$  is denoted by  $\text{Ann}(I)$ .

As we know, most properties of a ring are closely tied to the behavior of its ideals, so it is useful to study graphs or digraphs, associated to the ideals of a ring. To see an instance of these graphs, the reader is referred to [1, 2, 3, 5, 7, 8, 9]. The *regular digraph of ideals* of a ring  $R$ , denoted by  $\overrightarrow{\Gamma_{\text{reg}}}(R)$ , is a digraph whose vertex set is the set of all non-trivial ideals of  $R$  and for every two distinct vertices  $I$  and  $J$ , there is an arc from  $I$  to  $J$  if and only if  $I$  contains a  $J$ -regular element. The underlying graph of  $\overrightarrow{\Gamma_{\text{reg}}}(R)$ , denoted by  $\Gamma_{\text{reg}}(R)$ , is called the *regular graph* of ideals of  $R$ . For more information about this graph, see [3, 8]. The main aim of this paper is to prove that  $r(\Gamma_{\text{reg}}(R)) = 3$ .

## 2. Preliminary Results and Notation

In this section, the distance between any pair of vertices of  $\Gamma_{\text{reg}}(R)$  is determined, when  $R$  is an Artinian non-reduced ring and  $\Gamma_{\text{reg}}(R)$  is a connected graph.

**Remark 1.** Let  $I, J$  and  $K$  be three distinct vertices of  $\overrightarrow{\Gamma_{\text{reg}}}(R)$  and  $I \longrightarrow J \longrightarrow K$  be a directed path in  $\overrightarrow{\Gamma_{\text{reg}}}(R)$ . Then by using the definition, one can show that there is an arc from  $I$  to  $K$  in  $\overrightarrow{\Gamma_{\text{reg}}}(R)$ .

The following notations are used all over this paper.

**Notation.** Let  $I$  and  $K$  be (not necessarily distinct) ideals of  $R$ . We denote by  $\mathcal{C}^-(I, K)$ , the set of all non-trivial ideals  $J$  of  $R$  such that  $J$  contains an  $I$ -regular element and  $J$  contains a  $K$ -regular element. Also, we denote by  $\mathcal{C}^+(I, K)$ , the set of all non-trivial ideals  $J$  of  $R$  such that  $I$  contains a  $J$ -regular element and  $K$  contains a  $J$ -regular element. For simplicity, we denote  $\mathcal{C}^-(I, (0))$  and  $\mathcal{C}^+(I, R)$  by  $\mathcal{C}^-(I)$  and  $\mathcal{C}^+(I)$ , respectively.

**Remark 2.** Let  $R_1, \dots, R_n$  be rings,  $R \cong R_1 \times \dots \times R_n$  and  $I \cong I_1 \times \dots \times I_n$  and  $J \cong J_1 \times \dots \times J_n$  be two distinct vertices of  $\overrightarrow{\Gamma_{\text{reg}}}(R)$ . Then there is an arc from  $I$  to  $J$  if and only if  $I_i \in \mathcal{C}_R^-(J_i) \cup \{R_i\}$ , for every  $i$ , if and only if  $J_i \in \mathcal{C}_R^+(I_i) \cup \{(0)\}$ , for every  $i$ .

Assume that  $R$  is an Artinian ring such that  $\Gamma_{\text{reg}}(R)$  is a connected graph. Then by [8, Theorem 2.3],  $|\text{Max}(R)| \geq 3$  and  $R$  contains a field as its direct summand. So, [4, Theorem 8.7] implies that  $R \cong F_1 \times R_2 \times R_3$ , where  $F_1$  is a field,  $R_2$  is an Artinian local ring and  $R_3$  is an Artinian ring. Moreover, if  $R$  is non-reduced, then we also can suppose that  $R_3$  is not a field. Thus, any vertex of  $\Gamma_{\text{reg}}(R)$  belongs to one of the following subsets:

$$\mathfrak{A} = \{\mathfrak{a} = F_1 \times R_2 \times (0), \mathfrak{b} = F_1 \times (0) \times R_3, \mathfrak{d} = (0) \times R_2 \times R_3, \mathfrak{u} = (0) \times R_2 \times (0), \mathfrak{v} = (0) \times (0) \times R_3\};$$

$$\mathfrak{C} = \{\mathfrak{c} = F_1 \times I_2 \times I_3 \mid I_2 \times I_3 \neq R_2 \times R_3\} \setminus \{\mathfrak{a}, \mathfrak{b}\};$$

$$\mathfrak{W} = \{\mathfrak{w} = (0) \times K_2 \times K_3 \mid K_2 \times K_3 \neq (0)\} \setminus \{\mathfrak{d}, \mathfrak{u}, \mathfrak{v}\},$$

where  $I_2$  and  $K_2$  are ideals of  $R_2$  and  $I_3$  and  $K_3$  are ideals of  $R_3$ . From now, we use the above notations, to determine the distance between any disjoint pair of vertices of  $\Gamma_{reg}(R)$ . Also, for every Artinian ring  $R$ , we denote by  $n_F(R)$ , the number of fields appeared in the decomposition of  $R$  to a direct product of Artinian local rings.

**Proposition 3.** *Let  $R$  be an Artinian non-reduced ring. If  $\Gamma_{reg}(R)$  is a connected graph, then the following statements hold:*

- (1)  $d(\mathfrak{a}, \mathfrak{b}) = d(\mathfrak{u}, \mathfrak{v}) = d(\mathfrak{a}, \mathfrak{d}) = d(\mathfrak{b}, \mathfrak{d}) = 2$
- (2)  $d(\mathfrak{a}, \mathfrak{u}) = d(\mathfrak{b}, \mathfrak{v}) = d(\mathfrak{d}, \mathfrak{u}) = d(\mathfrak{d}, \mathfrak{v}) = d(\mathfrak{d}, \mathfrak{w}) = 1$ , where  $\mathfrak{w} \in \mathfrak{W}$ .
- (3)  $d(\mathfrak{a}, \mathfrak{v}) = d(\mathfrak{b}, \mathfrak{u}) = 3$ .
- (4) For every vertex  $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$ ,  $d(\mathfrak{a}, \mathfrak{c}) \leq 2$ . Moreover, the equality holds if and only if  $I_2 \neq R_2$  and  $I_3 \neq (0)$ .
- (5) For every vertex  $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$ ,  $d(\mathfrak{b}, \mathfrak{c}) \leq 2$ . Moreover, the equality holds if and only if  $I_2 \neq (0)$  and  $I_3 \neq R_3$ .
- (6) For every vertex  $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$ ,  $d(\mathfrak{u}, \mathfrak{w}) \leq 2$ . Moreover, the equality holds if and only if  $K_2 \neq R_2$  and  $K_3 \neq (0)$ .
- (7) For every vertex  $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$ ,  $d(\mathfrak{v}, \mathfrak{w}) \leq 2$ . Moreover, the equality holds if and only if  $K_2 \neq (0)$  and  $K_3 \neq R_3$ .

**Proof.** The assertions (1), (2) and (3) are clear and follow from the definition. Choose  $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$ . Clearly,  $\mathfrak{a}$  and  $\mathfrak{c}$  are adjacent if and only if either  $I_3 = (0)$  or  $I_2 = R_2$ . So, we can suppose that  $I_3 \neq (0)$  and  $I_2 \neq R_2$ . Since both of vertices  $\mathfrak{a}$  and  $\mathfrak{c}$  are adjacent to  $F_1 \times (0) \times (0)$ , we deduce that

$$d(\mathfrak{a}, \mathfrak{c}) = \begin{cases} 1; & \text{either } I_3 = (0) \text{ or } I_2 = R_2 \\ 2; & I_3 \neq (0) \text{ and } I_2 \neq R_2. \end{cases}$$

So, (4) follows. Also, the proofs of (5), (6) and (7) are similar to that of (4).  $\square$

**Proposition 4.** *Let  $R$  be an Artinian non-reduced ring. If  $\Gamma_{reg}(R)$  is a connected graph, then the following statements hold:*

- (1) For every two distinct vertices  $\mathfrak{c} = F_1 \times I_2 \times I_3$  and  $\mathfrak{c}' = F_1 \times J_2 \times J_3$  in  $\mathfrak{C}$ , we have  $d(\mathfrak{c}, \mathfrak{c}') \leq 2$ . Moreover,  $d(\mathfrak{c}, \mathfrak{c}') = 1$  if and only if  $I_2 = I_3 = (0)$ ,  $J_2 = J_3 = (0)$  or  $I_2 \times I_3$  and  $J_2 \times J_3$  are adjacent in  $\Gamma_{reg}(R_2 \times R_3)$ .
- (2) For every two distinct vertices  $\mathfrak{w} = (0) \times K_2 \times K_3$  and  $\mathfrak{w}' = (0) \times L_2 \times L_3$  in  $\mathfrak{W}$ , we have  $d(\mathfrak{w}, \mathfrak{w}') \leq 2$ . Moreover,  $d(\mathfrak{w}, \mathfrak{w}') = 1$  if and only if  $K_2 \times K_3 = R_2 \times R_3$ ,  $L_2 \times L_3 = R_2 \times R_3$  or  $K_2 \times K_3$  and  $L_2 \times L_3$  are adjacent in  $\Gamma_{reg}(R_2 \times R_3)$ .

**Proof.** This is a direct consequence of the definition.  $\square$

**Proposition 5.** Let  $R$  be an Artinian non-reduced ring. If  $\Gamma_{reg}(R)$  is a connected graph, then for every vertex  $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$ , we have:

- (1)  $d(\mathfrak{c}, \mathfrak{u}) = \begin{cases} 1; & I_2 = R_2 \\ 2; & I_2 \neq R_2 \text{ and } \mathcal{C}_{R_3}^-(I_3) \neq \emptyset. \\ 3; & \text{Otherwise.} \end{cases}$
- (2)  $d(\mathfrak{c}, \mathfrak{v}) = \begin{cases} 1; & I_3 = R_3 \\ 2; & I_3 \neq R_3 \text{ and either } I_2 = (0) \text{ or } \mathcal{C}^+(I_3) \neq \emptyset \\ 3; & \text{otherwise.} \end{cases}$
- (3)  $d(\mathfrak{c}, \mathfrak{d}) = \begin{cases} 2; & I_2 = R_2, I_3 = R_3 \text{ or } \mathcal{C}_{R_3}^+(I_3) \neq \emptyset \\ 3; & d(\mathfrak{c}, \mathfrak{d}) \neq 2 \text{ and } (I_2 = (0), I_3 = (0) \text{ or } \mathcal{C}_{R_3}^-(I_3) \neq \emptyset) \\ 4; & \text{Otherwise.} \end{cases}$
- (4)  $d(\mathfrak{a}, \mathfrak{w}) = \begin{cases} 1; & K_3 = (0) \\ 2; & K_3 \neq (0) \text{ and either } K_2 = R_2 \text{ or } \mathcal{C}^-(K_3) \neq \emptyset \\ 3; & \text{Otherwise.} \end{cases}$
- (5)  $d(\mathfrak{b}, \mathfrak{w}) = \begin{cases} 1; & K_2 = (0) \\ 2; & K_2 \neq (0) \text{ and } \mathcal{C}^+(K_3) \neq \emptyset \\ 3; & K_2 \neq (0) \text{ and } \mathcal{C}^+(K_3) = \emptyset. \end{cases}$

**Proof.** We only prove (3). The other assertions are proved, similarly. It is clear that  $\mathfrak{c}$  and  $\mathfrak{d}$  are not adjacent. On the other hand, since  $\mathfrak{u}$  and  $\mathfrak{d}$  are adjacent, (1) implies that  $2 \leq d(\mathfrak{c}, \mathfrak{d}) \leq 4$ . Now, we follow the proof in the following three steps:

Step 1.  $d(\mathfrak{c}, \mathfrak{d}) = 2$  if and only if  $I_2 = R_2$ ,  $I_3 = R_3$  or  $\mathcal{C}_{R_3}^+(I_3) \neq \emptyset$ :  
If  $I_2 = R_2$  ( $I_3 = R_3$ ), then both  $\mathfrak{c}$  and  $\mathfrak{d}$  are adjacent with  $\mathfrak{u}(\mathfrak{v})$ . Assume that  $J_3 \in \mathcal{C}_{R_3}^+(I_3)$ . Then  $\mathfrak{c}$  and  $\mathfrak{d}$  are adjacent to  $(0) \times (0) \times J_3$ . Therefore, in any case, we have  $d(\mathfrak{c}, \mathfrak{d}) = 2$ . Conversely, let  $d(\mathfrak{c}, \mathfrak{d}) = 2$ . Then by Remark 1, there exists a vertex  $J = J_1 \times J_2 \times J_3$  such that one of the paths  $\mathfrak{c} \longleftarrow J \longrightarrow \mathfrak{d}$  and  $\mathfrak{c} \longrightarrow J \longleftarrow \mathfrak{d}$  exists. By using Remark 2, one can deduce that the existence of the first path is impossible. So, we can suppose that only the second path exists. By Remark 2,  $J_1 = (0)$ . Hence either  $J_2 \neq (0)$  or  $J_3 \neq (0)$ . Therefore, Remark 2 implies that  $I_2 = R_2$ ,  $I_3 = R_3$  or  $J_3 \in \mathcal{C}^+(I_3) \neq \emptyset$ .

From now, suppose that  $I_2 \neq R_2$ ,  $I_3 \neq R_3$  and  $\mathcal{C}_{R_3}^+(I_3) = \emptyset$ . This means that  $d(\mathfrak{c}, \mathfrak{d}) \geq 3$ .

Step 2.  $d(\mathfrak{c}, \mathfrak{d}) = 3$  if and only if either  $I_2 = (0)$  or  $\mathcal{C}^-(I_3) \neq \emptyset$ :  
By Remark 1,  $d(\mathfrak{c}, \mathfrak{d}) = 3$  if and only if there exist two vertices, say  $J = J_1 \times J_2 \times J_3$  and  $L = L_1 \times L_2 \times L_3$ , such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow L_1 \times L_2 \times L_3 \longleftarrow (0) \times R_2 \times R_3 = \mathfrak{d}; \quad (1)$$

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow L_1 \times L_2 \times L_3 \longrightarrow (0) \times R_2 \times R_3 = \mathfrak{d}; \quad (2)$$

By Remark 2, Path (1) exists if and only if  $J_1 = F_1$  if and only if either  $J_2 \neq R_2$  or  $J_3 \neq R_3$ . On the other hand, from Remark 2, we deduce that  $J_2 \neq R_2$  if and only if  $I_2 = (0)$  and  $J_3 \neq R_3$  if and only if either  $I_3 = (0)$  or  $\mathcal{C}^-(I_3) \neq \emptyset$ . To complete the proof, it is enough to show that Path (2) does not exist. Since  $I_2 \neq R_2$  and  $\mathcal{C}^+(I_3) = \emptyset$ , we deduce that  $J_1 = J_2 = (0)$ . Thus Remark 2 implies that  $J = F_1 \times (0) \times (0)$  and  $L = (0) \times R_2 \times R_3$  and this contradicts the adjacency of  $J$  and  $L$ . Hence Path (2) does not exist and so, we are done.  $\square$

**Proposition 6.** *Let  $R$  be an Artinian non-reduced ring. If  $\Gamma_{reg}(R)$  is a connected graph, then for every  $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$  and every  $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$ , we have:*

$$d(\mathfrak{c}, \mathfrak{w}) = \begin{cases} 1; & I_2 \times I_3 \text{ contains a } K_2 \times K_3\text{-regular element} \\ 2; & I_2 = K_2 = (0), I_2 = K_2 = R_2 \text{ or } \mathcal{C}^-(I_3, K_3) \cup \mathcal{C}^+(I_3, K_3) \neq \emptyset \\ 5; & I_2, K_2 \text{ are nontrivial, } \mathcal{C}^+(I_3) \cup \mathcal{C}^-(I_3) \cup \mathcal{C}^+(K_3) \cup \mathcal{C}^-(K_3) = \emptyset \\ 3 \text{ or } 4; & \text{Otherwise.} \end{cases}$$

**Proof.** It is clear that  $\mathfrak{w}$  contains no  $\mathfrak{c}$ -regular element. Hence  $d(\mathfrak{c}, \mathfrak{w}) = 1$  if and only if  $\mathfrak{c}$  contains a  $\mathfrak{w}$ -regular element, say  $(x_1, x_2, x_3)$ , if and only if  $(x_2, x_3) \in I_2 \times I_3$  is a  $K_2 \times K_3$ -regular element. Assume that  $I_2 \times I_3$  contains no  $K_2 \times K_3$ -regular element. We claim that

$d(\mathfrak{c}, \mathfrak{w}) = 2$  if and only if  $I_2 = K_2 = (0)$ ,  $I_2 = K_2 = R_2$  or  $\mathcal{C}^-(I_3, K_3) \cup \mathcal{C}^+(I_3, K_3) \neq \emptyset$ . By Remark 1,  $d(\mathfrak{c}, \mathfrak{w}) = 2$  if and only if there exists a vertex, say  $J = J_1 \times J_2 \times J_3$ , such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow (0) \times K_2 \times K_3 = \mathfrak{w}; \quad (3)$$

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow (0) \times K_2 \times K_3 = \mathfrak{w}; \quad (4)$$

By Remark 2, Path (3) exists if and only if  $J_1 = F_1$  if and only if either  $J_2 \neq R_2$  or  $J_3 \neq R_3$ . On the other hand, from Remark 2, we deduce that  $J_2 \neq R_2$  if and only if  $I_2 = K_2 = (0)$  and  $J_3 \neq R_3$  if and only if  $\mathcal{C}^-(I_3, K_3) \neq \emptyset$ . Also, Path (4) exists if and only if  $J_1 = (0)$  if and only if either  $J_2 \neq (0)$  or  $J_3 \neq (0)$ . Moreover, Remark 2 implies that  $J_2 \neq (0)$  if and only if  $I_2 = K_2 = R_2$  and  $J_3 \neq (0)$  if and only if  $\mathcal{C}^+(I_3, K_3) \neq \emptyset$ . So, the claim is proved. Finally, assume that  $d(\mathfrak{c}, \mathfrak{w}) \geq 3$ ,  $I_2$  and  $K_2$  are non-trivial ideals of  $R_2$  and  $\mathcal{C}^-(I_3) \cup \mathcal{C}^+(I_3) \cup \mathcal{C}^-(K_3) \cup \mathcal{C}^+(K_3) = \emptyset$ . Then [8, Theorem 2.1] and Remark 1 imply that  $d_{\Gamma_{reg}(R)}^+(\mathfrak{w}) = d_{\Gamma_{reg}(R)}^-(\mathfrak{c}) = 0$ . We show that  $d(\mathfrak{c}, \mathfrak{w}) = 5$ . Suppose to the contrary,  $d(\mathfrak{c}, \mathfrak{w}) = 3$  or 4. Then by Remark 1, there exist three non-trivial ideals of  $R$ , say  $J$ ,  $L$  and  $P$  such that one of the following paths exists:

$$F_1 \times I_2 \times I_3 \longrightarrow J \longleftarrow L \longrightarrow (0) \times K_2 \times K_3; \quad (5)$$

$$F_1 \times I_2 \times I_3 \longleftarrow J \longrightarrow L \longleftarrow (0) \times K_2 \times K_3; \quad (6)$$

$$F_1 \times I_2 \times I_3 \longrightarrow J \longleftarrow L \longrightarrow P \longleftarrow (0) \times K_2 \times K_3; \quad (7)$$

$$F_1 \times I_2 \times I_3 \longleftarrow J \longrightarrow L \longleftarrow P \longrightarrow (0) \times K_2 \times K_3; \quad (8)$$

Since  $d_{\Gamma_{reg}(R)}^+(\mathfrak{w}) = d_{\Gamma_{reg}(R)}^-(\mathfrak{c}) = 0$ , Paths (6), (7) and (8) don't exist. Thus we can assume that Path (5) exists. Then Remark 2 implies that  $J = F_1 \times (0) \times (0)$  and  $L = (0) \times R_2 \times R_3$  which contradicts the adjacency of  $J$  and  $L$ . Therefore,  $d(\mathfrak{c}, \mathfrak{w}) = 5$  and the proof is complete.  $\square$

### 3. Main Results

In this section, it is proved that the radius of  $\Gamma_{reg}(R)$  equals 3. The central vertices are characterized, too. First we need some lemmas.

**Lemma 7.** *Let  $R$  be an Artinian ring which is not field. If  $n_F(R) \geq 1$ , then for every ideal  $I$  of  $R$ ,  $\mathcal{C}^+(I) \cup \mathcal{C}^-(I) \neq \emptyset$ .*

**Proof.** Since  $R$  is an Artinian ring, [4, Theorem 8.7] implies that  $R \cong F_1 \times R_2$ , where  $F_1$  is a field and  $R_2$  is an Artinian ring. For every ideal  $I = I_1 \times I_2$  of  $R$ , either  $I_1 = (0)$  or  $I_1 = F_1$ . If  $I_1 = (0)$ , then  $(0) \times R_2 \in \mathcal{C}^-(I)$ . Also, if  $I_1 = F_1$ , then  $F_1 \times (0) \in \mathcal{C}^+(I)$ . Thus in any case,  $\mathcal{C}^+(I) \cup \mathcal{C}^-(I) \neq \emptyset$ .  $\square$

Let  $R = R_1 \times R_2 \times \cdots \times R_n$  be an Artinian ring, where every  $R_i$  is an Artinian local ring. For every ideal  $I = I_1 \times I_2 \times \cdots \times I_n$  of  $R$ , setting

$$I_i^c = \begin{cases} R_i; & I_i = (0) \\ (0); & I_i = R_i \\ I_i; & I_i \text{ is a non-trivial ideal of } R_i, \end{cases}$$

we define the *complement* of  $I$  to be  $I^c = I_1^c \times I_2^c \times \cdots \times I_n^c$ . Also, for every subset  $X$  of ideals of  $R$ , by  $X^c$ , we mean the set  $\{I^c \mid I \in X\}$ .

**Lemma 8.** *Let  $R$  be an Artinian ring such that  $\Gamma_{reg}(R)$  is a connected graph. Then for every vertex  $I$  of  $\Gamma_{reg}(R)$ ,  $e(I) \geq 3$ .*

**Proof.** Let  $R$  be an Artinian ring. Since  $\Gamma_{reg}(R)$  is connected, [8, Theorem 3.2] and [4, Theorem 8.7] imply that  $R \cong F_1 \times R_2 \times \cdots \times R_n$ , where  $n \geq 3$ ,  $F_1$  is a field and every  $R_i$ ,  $2 \leq i \leq n$ , is an Artinian local ring. So,  $I = I_1 \times I_2 \times \cdots \times I_n$ , where every  $I_i$  is an ideal of  $R_i$ . We show that  $d(I, I^c) \geq 3$ . Suppose to the contrary,  $d(I, I^c) \leq 2$ . It is clear that  $I$  and  $I^c$  are not adjacent. Thus by Remark 1, there exists a vertex, say  $J = J_1 \times J_2 \times \cdots \times J_n$  such that one of the following paths exists:

$$I = I_1 \times I_2 \times \cdots \times I_n \longleftarrow J_1 \times J_2 \times \cdots \times J_n \longrightarrow I_1^c \times I_2^c \times \cdots \times I_n^c = I^c$$

$$I = I_1 \times I_2 \times \cdots \times I_n \longrightarrow J_1 \times J_2 \times \cdots \times J_n \longleftarrow I_1^c \times I_2^c \times \cdots \times I_n^c = I^c$$

If the first path exists, then Remark 2 implies that for every  $i$ ,  $J_i$  contains an  $I_i$ -regular element and  $J_i$  contains an  $I_i^c$ -regular element. Thus  $J_i = R_i$ , for every  $i$ , and hence  $J = R$ , a contradiction. Similarly, it is seen that the existence of the second path leads to a contradiction.  $\square$

**Theorem 9.** *Let  $R$  be an Artinian non-reduced ring such that  $\Gamma_{reg}(R)$  is a connected graph. Then the following statements hold:*

- (i)  $e(\mathfrak{a}) = e(\mathfrak{b}) = e(\mathfrak{u}) = e(\mathfrak{v}) = 3$ .
- (ii)  $e(\mathfrak{d}) = 3$  if and only if  $n_F(R) \geq 2$ .

**Proof.** (i) This follows from Propositions 3, 5 and Lemma 8.

(ii) By Proposition 3 and Lemma 8, it is enough to check  $d(\mathfrak{d}, \mathfrak{c})$ , where  $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$ . First suppose that  $n_F(R) \geq 2$ . If  $R_3$  is not field, then Lemma 7 yields that  $\mathcal{C}^+(I_3) \cup \mathcal{C}^-(I_3) \neq \emptyset$ , for every ideal  $I_3$  of  $R_3$ . Thus by Proposition 5 (3),  $d(\mathfrak{d}, \mathfrak{c}) \leq 3$ . Also, if  $R_3$  is a field, then  $I_3$  is a trivial ideal of  $R_3$  and so again by Proposition 5 (3),  $d(\mathfrak{d}, \mathfrak{c}) \leq 3$ . Hence  $e(\mathfrak{d}) = 3$ . Now, assume that  $n_F(R) = 1$ . Then Proposition 5 (3) implies that  $d(\mathfrak{d}, F_1 \times I_2 \times \text{Nil}(R_3)) = 4$  and so we are done.  $\square$

**Lemma 10.** *Let  $R$  be an Artinian ring such that  $\Gamma_{\text{reg}}(R)$  is connected. Then the following statements hold:*

- (i)  $\text{diam}(\Gamma_{\text{reg}}(R)) = 5$  if and only if  $n_F(R) = 1$ .
- (ii)  $\text{diam}(\Gamma_{\text{reg}}(R)) = 4$  if and only if  $n_F(R) = 2$ .
- (iii)  $\text{diam}(\Gamma_{\text{reg}}(R)) = 3$  if and only if  $n_F(R) \geq 3$ .

**Proof.** Since  $\Gamma_{\text{reg}}(R)$  is connected, [8, Theorem 2.1] implies that  $R \cong F_1 \times R_2 \times R_3$ , where  $F_1$  is a field,  $R_2$  is an Artinian local ring which is not field and  $R_3$  is an Artinian ring. Thus the assertion follows from [3, Theorem 2.10] and this fact that in any Artinian ring  $S$ ,  $Z(\text{Nil}(S)) = Z(S)$  if and only if  $S$  contains no field as its direct summand.  $\square$

From Lemmas 8 and 10, we have the following immediate corollary.

**Corollary 11.** *Let  $R$  be an Artinian non-reduced ring and  $\Gamma_{\text{reg}}(R)$  be a connected graph. If  $n_F(R) \geq 3$ , then for every vertex  $I$  of  $\Gamma_{\text{reg}}(R)$ ,  $e(I) = 3$ .*

**Lemma 12.** *Let  $I$  be an ideal of the Artinian ring  $R$ . Then  $\mathcal{C}^+(I) = \emptyset$  if and only if  $I \subseteq \text{Nil}(R)$ .*

**Proof.** First suppose that  $I \subseteq \text{Nil}(R)$ . We show that  $\mathcal{C}^+(I) = \emptyset$ . Suppose to the contrary  $J \in \mathcal{C}^+(I) \neq \emptyset$ . Then  $I$  contains a  $J$ -regular element, say  $x$ . Choose a non-zero element  $y \in J$ . Then there exists a positive integer  $n$  such that  $x^n y = 0$  and  $x^{n-1} y \neq 0$ . Since  $x^{n-1} y \in J$ , we deduce that  $x$  is not  $J$ -regular, a contradiction. Conversely, suppose that  $\mathcal{C}^+(I) = \emptyset$ . By [4, Theorem 8.7],  $R \cong R_1 \times R_2 \times \cdots \times R_n$ , where  $n$  is a positive integer and every  $R_i$  is an Artinian local ring. Thus  $I = I_1 \times I_2 \times \cdots \times I_n$ , where every  $I_i$  is an ideal of  $R_i$ . Since  $\mathcal{C}^+(I) = \emptyset$ , we deduce that every  $I_i$  is a proper ideal of  $R_i$ . Hence  $I \subseteq \text{Nil}(R)$ .  $\square$

According to Corollary 11, we only need to calculate the eccentricity of the vertices of  $R$ , when  $n_F(R) \leq 2$ . So from now, we focuss on a ring  $R$  which contains at most two fields as its direct summands.



**Theorem 13.** *Let  $R$  be an Artinian non-reduced ring and  $\Gamma_{reg}(R)$  be a connected graph. If  $n_F(R) \leq 2$  and  $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$  is a vertex of  $\Gamma_{reg}(R)$ , then the following statements hold:*

- (i) *If  $I_2 = R_2$ , then  $e(\mathfrak{c}) = 3$ .*
- (ii) *If  $I_2$  is a proper ideal of  $R_2$  and  $I_3 = (0)$ , then*

$$e(\mathfrak{c}) = \begin{cases} 3; & n_F(R) = 2 \text{ and either } (R_3 \text{ is not a field}) \text{ or } (R_3 \text{ is a field and } I_2 = (0)) \\ 4; & \text{Otherwise.} \end{cases}$$

- (iii) *If  $I_2 = (0)$  and  $I_3$  is a non-trivial ideal of  $R_3$ , then*

$$e(\mathfrak{c}) = \begin{cases} 3; & \text{either } n_F(R) = 2 \text{ or } I_3 \not\subseteq \text{Nil}(R_3) \\ 4; & n_F(R) = 1 \text{ and } I_3 \subseteq \text{Nil}(R_3). \end{cases}$$

- (iv) *If  $I_2$  is a non-trivial ideal of  $R_2$  and  $I_3 = R_3$ , then  $e(\mathfrak{c}) = 3$ .*
- (v) *Let  $I_2$  and  $I_3$  be non-trivial ideals of  $R_2$  and  $R_3$ , respectively. Then*

- (a) *If  $n_F(R) = 1$ , then  $e(\mathfrak{c}) = 3$  if and only if  $\mathcal{C}^+(I_3) \neq \emptyset$ .*
- (b) *If  $n_F(R) = 2$ , then  $R_3 \cong T_3 \times T_4 \times \cdots \times T_n$ , where  $T_3$  is a field and every  $T_i$ ,  $i \neq 4$ , is an Artinian local ring which is not field. Moreover,  $e(\mathfrak{c}) \neq 3$  if and only if  $I_3 = (0) \times Q_4 \times \cdots \times Q_n$ , where every  $Q_i$  is a non-trivial ideal of  $T_i$ .*

**Proof.** (i) Let  $\mathfrak{c} = F_1 \times R_2 \times I_3$ , where  $I_3$  is a non-trivial ideal of  $R_3$ . If  $x \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{u}, \mathfrak{v}\} \cup \mathfrak{C}$ , then Propositions 3, 4 and 5 imply that  $d(\mathfrak{c}, x) \leq 3$ . Also, the existence of the path

$$\mathfrak{c} = F_1 \times R_2 \times I_3 \longrightarrow (0) \times R_2 \times (0) \longleftarrow (0) \times R_2 \times R_3 \longrightarrow (0) \times K_2 \times K_3 = \mathfrak{w}$$

shows that  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ , for every vertex  $\mathfrak{w} \in \mathfrak{W}$ . Thus by Lemma 8,  $e(\mathfrak{c}) = 3$ .

- (ii) Let  $\mathfrak{c} = F_1 \times I_2 \times (0)$ , where  $I_2$  is a proper ideal of  $R_2$ . Then by Propositions 3, 4 and 5, we have  $d(\mathfrak{c}, x) \leq 3$ , for every vertex  $x \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{u}, \mathfrak{v}\} \cup \mathfrak{C}$ . Therefore, from Proposition 6, we deduce that  $3 \leq e(\mathfrak{c}) \leq 4$ . We follow the proof in the following cases:

Case 1.  $n_F(R) = 1$ . In this case,  $R \cong F_1 \times R_2 \times \cdots \times R_n$ , where  $n \geq 3$  and for every  $2 \leq i \leq n$ ,  $R_i$  is an Artinian local ring which is not a field. We prove that  $d(\mathfrak{c}, \text{Nil}(R)) = 4$ . Suppose to the contrary,  $d(\mathfrak{c}, \text{Nil}(R)) \neq 4$ . By Proposition 6,  $d(\mathfrak{c}, \text{Nil}(R)) = 3$ . Thus Remark 1 implies that there exist two vertices, say  $J = J_1 \times J_2 \times J_3$  and  $L = L_1 \times L_2 \times L_3$ , such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times I_2 \times (0) \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow L_1 \times L_2 \times L_3 \longleftarrow \text{Nil}(R) \quad (9)$$

$$\mathfrak{c} = F_1 \times I_2 \times (0) \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow L_1 \times L_2 \times L_3 \longrightarrow \text{Nil}(R) \quad (10)$$

By Lemma 12, Path (9) does not exist. So, we can assume that Path (10) exists. Thus Remark 2 implies that  $J = F_1 \times (0) \times (0)$  and  $L = (0) \times R_2 \times R_3$  which contradicts the adjacency of  $J$  and  $L$ . Therefore,  $e(\mathfrak{c}) = 4$ .

Case 2.  $n_F(R) = 2$ ,  $R_3$  is a field and  $I_2 \neq (0)$ . In this case, a similar proof to that of case 1 shows that  $d(\mathfrak{c}, (0) \times I_2 \times R_3) = 4$ . Thus, in this case,  $e(\mathfrak{c}) = 4$ .

Case 3.  $n_F(R) = 2$ ,  $R_3$  is a field and  $I_2 = (0)$ . Choose a vertex  $\mathfrak{w} \in \mathfrak{W}$ . Then there exists a non-trivial ideal  $K_2$  of  $R_2$  such that either  $\mathfrak{w} = (0) \times K_2 \times (0)$  or  $\mathfrak{w} = (0) \times K_2 \times R_3$ . If  $\mathfrak{w} = (0) \times K_2 \times (0)$ , then the vertex  $F_1 \times R_2 \times (0)$  is adjacent to both  $\mathfrak{c}$  and  $\mathfrak{w}$  and so  $d(\mathfrak{c}, \mathfrak{w}) = 2$ . Also, the existence of the path

$$\mathfrak{c} = F_1 \times (0) \times (0) \longleftarrow F_1 \times (0) \times R_3 \longrightarrow (0) \times (0) \times R_3 \longleftarrow (0) \times K_2 \times R_3 = \mathfrak{w}$$

implies that  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ . Thus, by Lemma 8,  $e(\mathfrak{c}) = 3$ .

Case 4.  $n_F(R) = 2$  and  $R_3$  is not a field. In this case,  $R_3 \cong T_3 \times T_4 \times \cdots \times T_n$ , where every  $T_i$ ,  $i \neq 3$ , is an Artinian local ring which is not field and  $T_3$  is a field. So, every vertex  $\mathfrak{w} \in \mathfrak{W}$  is of the form  $(0) \times Q_2 \times Q_3 \times \cdots \times Q_n$ . Now, in the following two subcases, we prove that  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ :

Subcase 1. There exists  $3 \leq j \leq n$  such that  $Q_j = T_j$ . With no loss of generality, one can assume that  $Q_3 = T_3$  and so, by Remark 2, the path

$$\mathfrak{c} \longleftarrow F_1 \times R_2 \times T_3 \times (0) \times \cdots \times (0) \longrightarrow (0) \times (0) \times T_3 \times (0) \times \cdots \times (0) \longleftarrow \mathfrak{w}$$

exists and hence  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ .

Subcase 2. For every  $3 \leq j \leq n$ ,  $Q_j \neq T_j$ . In this subcase,  $Q_3 = (0)$  and the existence of the path

$$\mathfrak{c} \longrightarrow F_1 \times (0) \times (0) \times \cdots \times (0) \longleftarrow F_1 \times R_2 \times (0) \times T_4 \times \cdots \times T_n \longrightarrow \mathfrak{w}$$

shows that  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ .

Therefore, in this case,  $e(\mathfrak{c}) = 3$  and this completes the proof of (ii).

(iii) Let  $\mathfrak{c} = F_1 \times (0) \times I_3$ , where  $I_3$  is a non-trivial ideal of  $R_3$ . Then by Propositions 3, 4 and 5, we only need to check  $d(\mathfrak{c}, \mathfrak{w})$ , where  $\mathfrak{w} \in \mathfrak{W}$ . Now, consider the following cases:

Case 1.  $n_F(R) = 1$  and  $I_3 \not\subseteq \text{Nil}(R_3)$ . In this case, Lemma 12 implies that  $\mathcal{C}^+(I_3) \neq \emptyset$ . Choose  $J_3 \in \mathcal{C}^+(I_3)$ . Then for every  $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$ , the path

$$\mathfrak{c} = F_1 \times (0) \times I_3 \longrightarrow (0) \times (0) \times J_3 \longleftarrow (0) \times R_2 \times R_3 \longrightarrow \mathfrak{w}$$

exists and so  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ , for every  $\mathfrak{w} \in \mathfrak{W}$ . Therefore, in this case,  $e(\mathfrak{c}) = 3$

Case 2.  $n_F(R) = 1$  and  $I_3 \subseteq \text{Nil}(R_3)$ . Since  $I_3 \subseteq \text{Nil}(R_3)$ , Lemma 12 implies that  $\mathcal{C}^+(I_3) = \emptyset$ . By [8, Theorem 8.7],  $R \cong F_1 \times R_2 \times \cdots \times R_n$ , where  $n \geq 3$  and for every  $2 \leq i \leq n$ ,  $R_i$  is an Artinian local ring which is not a field. Thus by [8, Theorem 2.1],  $\mathcal{C}^+(\text{Nil}(R_2 \times R_3 \times \cdots \times R_n)) \cup \mathcal{C}^-(\text{Nil}(R_2 \times R_3 \times \cdots \times R_n)) = \emptyset$ . We prove that  $d(\mathfrak{c}, \text{Nil}(R)) = 4$ . Suppose to the contrary,  $d(\mathfrak{c}, \text{Nil}(R)) \neq 4$ . By Proposition 6,  $d(\mathfrak{c}, \text{Nil}(R)) = 3$ . Thus Remark 1 implies that there exist two vertices, say  $J = J_1 \times J_2 \times J_3$  and  $L = L_1 \times L_2 \times L_3$ , such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times (0) \times I_3 \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow L_1 \times L_2 \times L_3 \longleftarrow \text{Nil}(R) \quad (11)$$

$$\mathfrak{c} = F_1 \times (0) \times I_3 \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow L_1 \times L_2 \times L_3 \longrightarrow \text{Nil}(R) \quad (12)$$

By Lemma 12, Path (11) does not exist. So, we can assume that Path (12) exists. Thus Remark 2 implies that  $J = F_1 \times (0) \times (0)$ . So,  $L = F_1 \times R_2 \times R_3$ , a contradiction. Thus  $d(\mathfrak{c}, \text{Nil}(R)) = 4$  and hence  $e(\mathfrak{c}) = 4$ .

Case 3.  $n_F(R) = 2$  and  $R_3$  is not a field. In this case,  $R_3 \cong T_3 \times T_4 \times \cdots \times T_n$ , where every  $T_i$ ,  $i \neq 3$ , is an Artinian local ring which is not field and  $T_3$  is a field. So, every vertex  $\mathfrak{w} \in \mathfrak{W}$  is of the form  $(0) \times Q_2 \times Q_3 \times \cdots \times Q_n$ . Now, in the following two subcases, we prove that  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ :

Subcase 1. There exists  $3 \leq j \leq n$  such that  $Q_j = T_j$ . With no loss of generality, one can assume that  $Q_3 = T_3$  and so, by Remark 2, the path

$$\mathfrak{c} \longleftarrow F_1 \times (0) \times T_3 \times T_4 \times \cdots \times T_n \longrightarrow (0) \times (0) \times T_3 \times (0) \times \cdots \times (0) \longleftarrow \mathfrak{w}$$

exists and hence  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ .

Subcase 2. For every  $3 \leq j \leq n$ ,  $Q_j \neq T_j$ . In this subcase, the existence of the path

$$\mathfrak{c} \longrightarrow F_1 \times (0) \times (0) \times \cdots \times (0) \longleftarrow F_1 \times (0) \times T_3 \times \cdots \times T_n \longrightarrow \mathfrak{w}$$

shows that  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ .

Therefore, in this case, Lemma 8 implies that  $e(\mathfrak{c}) = 3$ .

(iv) Assume that  $\mathfrak{c} = F_1 \times I_2 \times R_3$ , where  $I_2$  is a non-trivial ideal of  $R_2$ . By Propositions 3, 4 and 5, we have  $d(\mathfrak{c}, x) \leq 3$ , for every vertex  $x \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{u}, \mathfrak{v}\} \cup \mathfrak{C}$ . Also, for every vertex  $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$ , the path

$$\mathfrak{c} \longrightarrow (0) \times (0) \times R_3 \longleftarrow (0) \times R_2 \times R_3 \longrightarrow \mathfrak{w}$$

exists and hence by Lemma 8,  $e(\mathfrak{c}) = 3$ .

(v) Let  $\mathfrak{c} = F_1 \times I_2 \times I_3$ , where  $I_2$  and  $I_3$  are non-trivial ideals of  $R_2$  and  $R_3$ , respectively. Then by Propositions 3,4 and 5, we have  $d(\mathfrak{c}, x) \leq 3$ , for every vertex  $x \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{u}, \mathfrak{v}\} \cup \mathfrak{C}$ . Hence by Lemma 8,  $e(\mathfrak{c})$  depends only on  $d(\mathfrak{c}, \mathfrak{d})$  and  $d(\mathfrak{c}, \mathfrak{w})$ , where  $\mathfrak{w} \in \mathfrak{W}$ . On the other hand, Proposition 5(2) implies that  $d(\mathfrak{c}, \mathfrak{d}) \leq 3$  if and only if  $\mathcal{C}^+(I_3) \cup \mathcal{C}^-(I_3) \neq \emptyset$ . Therefore, by Lemma 8,  $e(\mathfrak{c}) = 3$  if and only if  $\mathcal{C}^+(I_3) \cup \mathcal{C}^-(I_3) \neq \emptyset$  and  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ , for every  $\mathfrak{w} \in \mathfrak{W}$ . To complete the proof, we consider the following three cases:

Case 1.  $\mathcal{C}^+(I_3) \neq \emptyset$ . In this case, choose  $J_3 \in \mathcal{C}^+(I_3)$ . Then for every  $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$ , the path

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow (0) \times (0) \times J_3 \longleftarrow (0) \times R_2 \times R_3 \longrightarrow (0) \times K_2 \times K_3 = \mathfrak{w}$$

exists and so  $e(\mathfrak{c}) = 3$ .

Case 2.  $\mathcal{C}^+(I_3) = \emptyset$  and  $n_F(R) = 1$ . In this case, we prove that  $d(\mathfrak{c}, (0) \times I_2 \times \text{Nil}(R_3)) \geq 4$ . Suppose to the contrary,  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ . Then Remark 1 implies that there are two vertices, say  $J = J_1 \times J_2 \times J_3$  and  $L = L_1 \times L_2 \times L_3$ , such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow L_1 \times L_2 \times L_3 \longleftarrow (0) \times I_2 \times \text{Nil}(R_3) \quad (13)$$

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow L_1 \times L_2 \times L_3 \longrightarrow (0) \times I_2 \times \text{Nil}(R_3) \quad (14)$$

By Lemma 12, Path (13) does not exist. So, we can assume that Path (14) exists. Thus Remark 2 implies that  $J = F_1 \times (0) \times (0)$  and  $L = (0) \times R_2 \times R_3$  which contradicts the adjacency of  $J$  and  $L$ . Therefore,  $e(\mathfrak{c}) = 4$ .

Case 3.  $\mathcal{C}^+(I_3) = \emptyset$  and  $n_F(R) = 2$ . In this case, Lemma 7 implies that  $\mathcal{C}^-(I_3) \neq \emptyset$ . Since  $I_3$  is a non-trivial ideal of  $R_3$  and  $n_F(R) = 2$ , we deduce that  $R_3 \cong T_3 \times T_4 \times \cdots \times T_n$ , where  $T_3$  is a field and for every  $4 \leq i \leq n$ ,  $T_i$  is an Artinian local ring which is not field. So,  $I_3 = (0) \times Q_4 \times \cdots \times Q_n$ , where every  $Q_i$  is a proper ideal of  $T_i$ . To complete the proof, we consider the following two subcases:

Subcase 1. Every  $Q_j$ ,  $4 \leq j \leq n$ , is a non-trivial ideal of  $T_j$ . In this subcase, we prove that  $d(\mathfrak{c}, (0) \times I_2 \times I_3^c) \geq 4$  (Note that  $I_3^c = T_3 \times Q_4 \times \cdots \times Q_n$ ). Suppose to the contrary,  $d(\mathfrak{c}, (0) \times I_2 \times I_3^c) \leq 3$ . Then by Remark 1, there are two vertices, say  $J = J_1 \times J_2 \times J_3$  and  $L = L_1 \times L_2 \times L_3$ , such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow L_1 \times L_2 \times L_3 \longleftarrow (0) \times I_2 \times I_3^c \quad (15)$$

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow L_1 \times L_2 \times L_3 \longrightarrow (0) \times I_2 \times I_3^c \quad (16)$$

If Path (15) exists, then by Remark 2,  $J = F_1 \times R_2 \times (0) \times T_4 \times \cdots \times T_n$  and  $L = (0) \times (0) \times T_3 \times (0) \times \cdots \times (0)$  and this contradicts the adjacency of  $J$  and  $L$ . Also, since  $\mathcal{C}^+(I_3) = \emptyset$ , by Remark 2, the existence of the Path (16) implies that  $J = F_1 \times (0) \times (0)$  and  $L = (0) \times R_2 \times R_3$ , a contradiction. Hence  $e(\mathfrak{c}) \geq 4$

Subcase 2. There exists  $4 \leq j \leq n$  such that  $Q_j = (0)$ . With no loss of generality, one can assume that  $Q_4 = (0)$ . We show that for every  $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$ ,  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ . From Lemma 7, we deduce that  $\mathcal{C}^+(K_3) \cup \mathcal{C}^-(K_3) \neq \emptyset$ . If  $\mathcal{C}^-(K_3) \neq \emptyset$ , then there exists a non-trivial ideal  $L_3 \in \mathcal{C}^-(K_3)$ . Thus the path

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow F_1 \times (0) \times (0) \longleftarrow F_1 \times R_2 \times L_3 \longrightarrow (0) \times K_2 \times K_3 = \mathfrak{w}$$

exists and so there is no thing to prove. Thus we can suppose that  $\mathcal{C}^-(K_3) = \emptyset$  and so  $K_3 = T_3 \times Q_4 \times \cdots \times Q_n$ , where  $Q_i \neq (0)$ , for every  $4 \leq i \leq n$ . Setting  $J_3 = T_3 \times (0) \times T_5 \times \cdots \times T_n$  and  $L_3 = T_3 \times (0) \times \cdots \times (0)$ , Remark 2 implies that the path

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longleftarrow F_1 \times R_2 \times J_3 \longrightarrow (0) \times (0) \times L_3 \longleftarrow (0) \times K_2 \times K_3 = \mathfrak{w}$$

exists and so  $d(\mathfrak{c}, \mathfrak{w}) \leq 3$ . Therefore, in this subcase,  $e(\mathfrak{c}) = 3$ . So, we are done.  $\square$

Let  $R$  be an Artinian ring and  $I$  and  $J$  be two non-trivial ideals of  $R$ . Then it is clear that  $I \in \mathcal{C}^+(J)$  if and only if  $I^c \in \mathcal{C}^-(J^c)$ . Moreover,  $I \longrightarrow J$  is an arc of  $\overrightarrow{\Gamma_{reg}}(R)$  if and only if  $J^c \longrightarrow I^c$  of  $\overrightarrow{\Gamma_{reg}}(R)$ . Thus  $d(I, J) = d(I^c, J^c)$  and hence  $e(I) = e(I^c)$ . Moreover, we have  $I \in \mathfrak{C}$  if and only if  $I^c \in \mathfrak{W}$ . Thus using this facts and applying the similar proof to that of Theorem 13, one can prove the following theorem.

**Theorem 14.** *Let  $R$  be an Artinian non-reduced ring such that  $\Gamma_{reg}(R)$  be a connected graph. If  $n_F(R) \leq 2$  and  $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$  is a vertex of  $\Gamma_{reg}(R)$ , then the following statements hold:*

- (i) *If  $K_2 = (0)$ , then  $e(\mathfrak{w}) = 3$ .*
- (ii) *If  $K_2$  is a non-trivial ideal of  $R_3$  and  $K_3 = R_3$ , then*

$$e(\mathfrak{w}) = \begin{cases} 3; & n_F(R) = 2 \text{ and } R_3 \text{ is not a field} \\ 4; & \text{Otherwise.} \end{cases}$$

- (iii) *If  $K_2 = R_2$ , then*

$$e(\mathfrak{w}) = \begin{cases} 3; & \text{Either } n_F(R) = 2 \text{ or } \mathcal{C}^-(K_3) \neq \emptyset \\ 4; & \text{Otherwise.} \end{cases}$$

- (iv) *If  $K_2$  is a non-trivial ideal of  $R_2$  and  $K_3 = (0)$ , then  $e(\mathfrak{w}) = 3$ .*
- (v) *Let  $K_2$  and  $K_3$  be non-trivial ideals of  $R_2$  and  $R_3$ , respectively. Then*

- (a) If  $n_F(R) = 1$ , then  $e(\mathfrak{w}) = 3$  if and only if  $\mathcal{C}^-(K_3) \neq \emptyset$ .
- (b) If  $n_F(R) = 2$ , then  $R_3 \cong T_3 \times T_4 \times \cdots \times T_n$ , where  $T_3$  is a field and every  $T_i$ ,  $i \neq 4$ , is an Artinian local ring which is not field. Moreover,  $e(\mathfrak{w}) \neq 3$  if and only if  $K_3 = T_3 \times Q_4 \times \cdots \times Q_n$ , where every  $Q_i$  is a non-trivial ideal of  $T_i$ .

Finally, we prove the main theorem of this paper.

**Theorem 15.** *If  $R$  is an Artinian ring and  $\Gamma_{reg}(R)$  is connected, then  $r(\Gamma_{reg}(R)) = 3$ .*

**Proof.** Let  $\Gamma_{reg}(R)$  be a connected graph. Then [8, Theorem 2.3] implies that  $|\text{Max}(R)| \geq 3$  and  $R$  contains a field as its direct summand. Since  $R$  has at least three maximal ideals, [4, Theorem 8.7] implies that  $R \cong F_1 \times R_2 \times R_3$ , where  $F_1$  is a field and  $R_2$  and  $R_3$  are Artinian rings. First suppose that  $R$  is reduced. Then  $R \cong F_1 \times \cdots \times F_n$ , where  $n \geq 3$  and every  $F_i$  is a field. For every ideal  $I = I_1 \times \cdots \times I_n$  of  $R$ , define

$$\Delta_I = \{k \mid 1 \leq k \leq n \text{ and } I_k = F_k\}$$

and

$$\Omega = \{\Delta_I \mid I \text{ is a non-trivial ideal of } R\}.$$

Clearly,  $\Delta_I = \Delta_J$  if and only if  $I = J$ . Thus there is a one to one correspondence between  $\Omega$  and the set of proper and non-empty subsets of  $\{1, \dots, n\}$ . We claim that  $d(I, J) \leq 3$ , for every two distinct vertices  $I$  and  $J$  of  $\Gamma_{reg}(R)$ . To see this, we consider the following two cases:

Case 1.  $\Delta_I \cap \Delta_J = \emptyset$ . Since  $n \geq 3$ , by pigeon-hole principle and with no loss of generality, we can assume that  $|\Delta_I| \geq |\Delta_J|$ ,  $|\Delta_I| \geq 2$  and  $1 \in \Delta_I$ . Let  $I_1$  and  $I_2$  be two vertices of  $\overrightarrow{\Gamma_{reg}}(R)$  such that  $\Delta_{I_1} = \{1\}$  and  $\Delta_{I_2} = \Delta_J \cup \{1\}$ . Then there is the path  $I \longrightarrow I_1 \longleftarrow I_2 \longrightarrow J$  in  $\overrightarrow{\Gamma_{reg}}(R)$  and hence  $d(I, J) \leq 3$ .

Case 2.  $\Delta_I \cap \Delta_J \neq \emptyset$ . If either  $\Delta_I \subset \Delta_J$  or  $\Delta_J \subset \Delta_I$ , then  $I$  and  $J$  are adjacent. So, we can assume that neither  $\Delta_I \not\subseteq \Delta_J$  nor  $\Delta_J \not\subseteq \Delta_I$ . Choose  $i \in \Delta_I \cap \Delta_J$ . Then it is clear that  $I_i = (0) \times \cdots \times F_i \times \cdots \times (0)$  is adjacent to both  $I$  and  $J$  and so  $d(I, J) = 2$ .

So, the claim is proved. Thus from Lemma 8, we deduce that  $e(I) = 3$ . Therefore, in any case,  $r(\Gamma_{reg}(R)) = 3$ . Now, suppose that  $R$  is non-reduced. Then the assertion follows from Theorems 9, 13, 14 and Corollary 11.  $\square$

Using Theorems 9, 13 and 14, we have the following immediate corollary in which the center of  $\Gamma_{reg}(R)$  is determined.

**Corollary 16.** *Let  $R$  be an Artinian non-reduced ring and  $\Gamma_{reg}(R)$  be a connected graph. Then the following statements hold:*

- (i) *If  $n_F(R) = 1$  and  $X = \{F_1 \times I_2 \times I_3 \mid \text{either } I_2 = R_2 \text{ or } I_3 \not\subseteq \text{Nil}(R_3)\}$ , then the center of  $\Gamma_{reg}(R)$  equals  $X \cup X^c$ .*
- (ii) *If  $n_F(R) = 2$  and  $|\text{Max}(R)| = 3$ , then every vertex of  $\Gamma_{reg}(R)$  is central.*
- (iii) *If  $n_F(R) = 2$  and  $|\text{Max}(R)| \geq 4$ , then  $R \cong F_1 \times T_2 \times F_3 \times T_4 \times \cdots \times T_n$ , where  $F_1$  and  $F_3$  are fields and every  $T_i$  is an Artinian local ring which is not field. Moreover, the vertex  $I$  is central if and only if neither  $I = F_1 \times I_2 \times (0) \times I_4 \times \cdots \times I_n$  nor  $I = (0) \times I_2 \times F_3 \times I_4 \times \cdots \times I_n$ , for every non-trivial ideals  $I_i$  of  $T_i$ .*
- (iv) *If  $n_F(R) \geq 3$ , then every vertex of  $\Gamma_{reg}(R)$  is central.*

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